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EQUIVALENCE OF KÄHLER MANIFOLDS AND OTHER EQUIVALENCE PROBLEMS

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1. Let M be a complex manifold, $\dim_C M = n$, τ be its tangent bundle and and J be the canonical endomorphism of τ such that $J^2 = -1$. Considering J as multiplication by $\sqrt{-1}$ we can give τ a structure of a complex vector bundle. A complex 1-dimensional subspace of τ is called a *holomorphic* 2-plane. The holomorphic 2-planes form a bundle $G_1(M, C)$ on M, whose fiber is $(n - 1)$-dimensional complex projective space; it is a subbundle of the grassmann bundle $G_2(M, \boldsymbol{R})$ of 2-planes on M. Let g be a hermitian metric on M, i.e., a Riemann metric for which J is an isometry. Let K be the corresponding sectional curvature which is a real valued function on $G_2(M, \boldsymbol{R})$. The restriction of K to $G_1(M, C)$ is called the *holomorphic curvature*, and denoted by H. Since H comes quite canonically from the metric and the complex structure, a natural question arises: is *H characteristic of the complex geometry*? We shall show that this is so in a certain sense for Kähler manifolds.

Theorem 1. *Let M, M^* be connected Kähler manifolds with corresponding holomorphic curvature functions H, H^* respectively. Suppose that* $\dim_C M \geq 2$ *and there exists a diffeomorphism $f: M \to M^*$ such that*

$$(*) \qquad f^*H^* = H .$$

Then either (i) *$H = H^* =$ constant and hence M, M^* are locally holomorphically isometric or* (ii) *f is a holomorphic or antiholomorphic isometry.*

Note that $(*)$ implicitly requires that f carries $G_1(M, C)$ into $G_1(M^*, C)$, which is a strong restriction. Note also that when $H = H^* =$ constant, then $(*)$ is redundant and the conclusion in (i) is classical.

We have not considered the nonKähler case, but the restriction to Kähler manifolds seems natural for the following reason. A Riemann manifold *at a point* can be approximated by a Euclidean space up to the *first* order ("existence of geodesic coordinates"), and curvature is precisely the second order effect which carries a substantial information about local geometry. Analogously among complex manifolds with hermitian metric, Kähler manifolds are characterized by the property of existence of holomorphic geodesic coordinates. For this reason it is only in the Kähler case that one may expect H to carry ade-

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quate information. This may explain why holomorphic curvature has seldom helped in the problems concerning non-Kähler manifolds.

The above formulation in terms of H-preserving maps is one type of formulation of the so-called "equivalence problem". The general equivalence problem for G-structure was posed by E. Cartan. In this general setup one has no natural nonlinear "curvature functions", so the problem is formulated in terms of Pfaffians, and the solution of the problem contains complete local information about a G-structure. E. Cartan also considered special equivalence problems for specific geometric structures, e.g., in [1] he studied congruence of surfaces in $\boldsymbol{R}^3$ under the hypothesis that they have a given second fundamental form. See the references in [3] for later development. I. Singer [8], K. Nomizu and K. Yano [7] have considered equivalence of Riemann manifolds. Perhaps because of the general philosophy introduced in the general equivalence problem for G-structures the idea of considering certain multilinear tensors runs through all these works. Our contention is simply that in the situation of geometric interest where curvature functions are available, formulation of the problem in terms of these functions leads to elegant and appealing solutions. The method works because the functions are *nonlinear*. It has a drawback that for it to work interesting curvature functions should be available. Secondly going from the fundamental structure tensors to such curvature functions one loses the analytic content somewhere and the generic hypothesis are *unavoidable*. This is not quite apparent in Theorem 1 because the inherent rigidity of the Kähler structure is helpful. But this will be apparent from the discussions in [3], [4]. Yau [9] has given an example of 3-dimensional manifolds (with nonisotropic points) where a sectional curvature preserving diffeomorphism is not an isometry. Similarly it is possible to construct examples of manifolds in all dimensions where the generalized Schur's theorem does not hold without some further hypothesis; cf. [5]. In this connection we mention important results of T. Nasu [6] and S. T. Yau [9] concerning diffeomorphisms preserving sectional and Ricci curvatures.

Finally here is a table of "good" curvature functions which have certain basic similarities:

	Object	*Curvatnre function*
1)	Hypersurfaces in a space of constant curvature	nomal curvature
2)	Riemann manifolds	sectional curvature
3)	Kähler manifolds	holomorphic curvature
4)	Conformally flat manifolds	Ricci curvature

It is helpful to keep these analogies in mind while formulating problems involving such curvature functions, although the techniques in general will change from case to case.

2. We begin with the proof of Theorem 1. Let us first recall that the Kähler metric is characterized by the property

$$\nabla J = 0 , \tag{1}$$

where ∇ is the connection defined by the metric. This implies

$$R(x, y)J = 0 , \tag{2}$$

where R is the curvature tensor considered as a derivation of the tensor algebra. (2) and the usual properties of the curvature tensor imply

$$\langle R(x, y)Jz, Jw\rangle = \langle R(x, y)z, w\rangle = \langle R(Jx, Jy)z, w\rangle . \tag{3}$$

Let $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ be a locally defined orthonormal frame on M. We shall call a frame of this type an *adapted* frame on the manifold, and write $\bar{e}_i$ for Je_i. As a rule the indices i, j, k range from 1 to n and refer to $e_i, e_j, e_k, \cdots$. $\bar{i}, \bar{j}, \bar{k}$ will refer to $\bar{e}_i, \bar{e}_j, \bar{e}_k$, etc. Property (3) in terms of the components of R reads, e.g.,

$$R_{ij,kl} = R_{ij,\bar{k}\bar{l}} = R_{\bar{i}\bar{j},kl} . \tag{4}$$

If the holomorphic curvature is constant c, then

(5) $R_{i\bar{i}i\bar{i}} = c$, $R_{i\bar{i}j\bar{j}} = \frac{1}{2}c \ (i \neq j)$, $R_{ijij} = \frac{1}{4}c \ (i \neq j)$, and the remaining components which cannot be reduced to this form are zero due to (4) or the usual properties of the curvature tensor.

Conversely (5) implies that the holomorphic curvature is constant.

We shall say that a point $p \in M$ is *isotropic* with respect to H if H has the same value for all holomorphic 2-planes at p.

Lemma 1. *Suppose $p \in M$ is not isotropic. Then there exists an adapted frame such that*

$$(R_{i\bar{i}i\bar{j}})^2 + (R_{i\bar{i}i\bar{i}} - R_{j\bar{j}j\bar{j}})^2 \neq 0 \tag{$*$}$$

for all (i, j), $i \neq j$.

This proof requires a "general position" argument and is very much similar to Lemma 2, § 1 of [2]. The crucial case is when $\dim_C M = 2$. The proof will be left to the reader.

3. Let now $f: M \to M^*$ be an H-preserving diffeomorphism of Kähler manifolds of complex dimension ≥ 2. For this to make sense we require of course that

$$f_*(G_1(M, C)) \subseteq G_1(M^*, C) .$$

Lemma 2. *f is holomorphic or antiholomorphic.*

Proof. We have to show that either $f_*J = Jf_*$ or $f_*J = -Jf_*$. The crucial case is when $\dim_C M = 2$. So we assume $\dim_C M = 2$ and leave the general case to the reader. Let $\{e_1, e_2, Je_1, Je_2\}$ be an adapted frame at $p \in M$. Let

$$f_*e_i = e_i^* , \quad f_*Je_i = h_i^* \qquad i = 1, 2 .$$

Since

$$f_*\{e_i, Je_i\} = \{e_i^*, h_i^*\} , \qquad i = 1, 2 ,$$

is a holomorphic 2-plane, we have

$$(1) \qquad h_i^* = a_i e_i^* + b_i J e_i^* , \qquad i = 1, 2 .$$

Also for all $(x, y) \neq (0, 0)$, $x, y \in \mathbf{R}$,

$$f_*\{xe_1 + ye_2, xJe_1 + yJe_2\} = \{xe_1^* + ye_2^*, xh_1^* + yh_2^*\}$$

is a holomorphic 2-plane. Hence

$$\begin{aligned} xh_1^* + yh_2^* &= x(a_1e_1^* + b_1Je_1^*) + y(a_2e_2^* + b_2Je_2^*) \\ &= \lambda(xe_1^* + ye_2^*) + \mu(xJe_1^* + yJe_2^*) \end{aligned}$$

for suitable λ, μ. This implies $a_1 = \lambda = a_2$, $b_1 = \mu = b_2$. Writing $a_1 = a_2 = a$, $b_1 = b_2 = a$ we have

$$(1)' \qquad h_i^* = ae_i^* + bJe_i^* , \qquad i = 1, 2 .$$

Again for all $(x, y) \neq (0, 0)$, $x, y \in \mathbf{R}$,

$$f_*\{xe_1 + yJe_2, xJe_1 - ye_2\} = \{xe_1^* + yh_2^*, xh_1^* - ye_2^*\}$$

is a holomorphic 2-plane. So

$$\begin{aligned} xh_1^* - ye_2^* &= x(ae_1^* + bJe_1^*) - ye_2^* \\ &= \lambda\{xe_1^* + y(ae_2^* + bJe_2^*)\} + \mu\{xJe_1^* + y(aJe_2^* - be_2^*)\} \end{aligned}$$

for suitable λ, μ. This implies

$$(2) \quad a = \lambda,\ b = \mu , \quad \lambda a - \mu b = a^2 - b^2 = -1 , \quad \lambda b + \mu a = 2ab = 0 .$$

The only solutions of these equations are $a = 0$, $b = \pm 1$. $b = 1$ corresponds to $f_*J = Jf_*$, and $b = -1$ to $f_*J = -Jf_*$.

4. Considering M^* or its complex conjugate we shall assume for definiteness that f is holomorphic. Let $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ be an adapted frame on the manifold. Let $f_*e_i = e_i^*$. Since f is holomorphic, we have $f_*Je_i = Je_i^*$. Let

$$(1)\qquad \|e_i^*\|^{*2} = a_i = \|Je_i^*\|^{*2}\,, \quad \langle e_i^*, e_j^*\rangle^* = a_{ij} = \langle Je_i^*, Je_j^*\rangle^*\,, \quad i \neq j\,.$$

The crucial lemma is

Lemma 3. *If $p \in M$ is nonisotropic, then f_{*p} is a homothety.*

Proof. In the notation introduced above we have to show

$$a_i = a_j \quad \text{and} \quad a_{ij} = 0 \qquad i \neq j\,.$$

Since f is H-preserving, we have for all $(x, y) \neq (0, 0)$, $x, y \in \boldsymbol{R}$,

$$(2)\qquad \begin{aligned} &\langle R(xe_i + ye_j, x\bar{e}_i + y\bar{e}_j)xe_i + ye_j, x\bar{e}_i + y\bar{e}_j\rangle/(x^2 + y^2)^2 \\ &= \frac{\langle R^*(xe_i^* + ye_j^*, x\bar{e}_i^* + y\bar{e}_j^*)xe_i^* + ye_j^*, x\bar{e}_i^* + y\bar{e}_j^*\rangle^*}{(a_ix^2 + 2a_{ij}xy + a_jy^2)^2}\,. \end{aligned}$$

Let the numerator of the left hand side be

$$\mathrm{Num} = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4\,,$$

where

$$(3)\qquad \begin{aligned} &A = R_{i\bar{i}i\bar{i}}\,, \quad E = R_{j\bar{j}j\bar{j}}\,, \quad B = 4R_{i\bar{i}i\bar{j}}\,, \quad D = 4R_{j\bar{j}j\bar{i}}\,, \\ &C = 2(R_{i\bar{i}j\bar{j}} + 2R_{i\bar{j}i\bar{j}})\,. \end{aligned}$$

In (3) we have used the Kähler property. Similarly we write the numerator of the right hand side as

$$\mathrm{Num}^* = A^*x^4 + B^*x^3y + C^*x^2y^2 + D^*xy^3 + E^*y^4\,,$$

where $A^*, B^* \cdots$ are obtained by $*$-ing (3). (2) leads to the following equations:

$$(4)\qquad A^* = a_i^2A\,, \qquad E^* = a_j^2E\,;$$

$$(5)\qquad B^* = a_i^2B + 4a_ia_{ij}A\,, \qquad D^* = a_j^2D + 4a_ja_{ij}E\,;$$

$$(6)\qquad \begin{aligned} 2A^* + C^* &= 2(a_ia_j + 2a_{ij}^2)A + 4a_ia_{ij}B + a_i^2C\,, \\ 2E^* + C^* &= 2(a_ia_j + 2a_{ij}^2)E + 4a_ja_{ij}D + a_j^2C\,; \end{aligned}$$

$$(7)\qquad \begin{aligned} 2B^* + D^* &= 4a_ja_{ij}A + 2(a_ia_j + 2a_{ij}^2)B + 4a_ia_{ij}C + a_i^2D\,, \\ 2D^* + B^* &= 4a_ia_{ij}E + 2(a_ia_j + 2a_{ij}^2)D + 4a_ja_{ij}C + a_j^2B\,; \end{aligned}$$

$$(8)\qquad \begin{aligned} A^* + 2C^* + E^* &= a_j^2A + 4a_ja_{ij}B \\ &\quad + 2(a_ia_j + 2a_{ij}^2)C + 4a_ia_{ij}D + a_i^2E\,. \end{aligned}$$

Substituting (4) in (6) we obtain

$$
(9)\qquad \begin{aligned} C^* &= (4a_{ij}{}^2 + 2a_ia_j - 2a_i{}^2)A + 4a_ia_{ij}B + a_i{}^2C \\ &= (4a_{ij}{}^2 + 2a_ia_j - 2a_j{}^2)E + 4a_ja_{ij}D + a_j{}^2C \end{aligned}
$$

and putting this in (8) we get

$$
(10)\qquad \{4a_{ij}{}^2 - (a_i - a_j)^2\}(A + E - C) + 4(a_i - a_j)a_{ij}(B - D) = 0 .
$$

Similarly, substituting the values of B^* and D^* in (7) and subtracting the first equation in (7) from the second we obtain

$$
(11)\qquad 4(a_i - a_j)a_{ij}(A + E - C) - \{4a_{ij}{}^2 - (a_i - a_j)^2\}(B - D) = 0 .
$$

From (10) and (11) we conclude that if $(A + E - C)^2 + (B - D)^2 \neq 0$, then

$$
(12)\qquad 4a_{ij}{}^2 - (a_i - a_j)^2 = 0 , \qquad (a_i - a_j)a_{ij} = 0 ,
$$

which clearly imply that $a_i = a_j$ and $a_{ij} = 0$. It remains to consider the case when $(A + E - C) = 0$ and $(B - D) = 0$. In this case (9) implies

$$
(13)\qquad \{4a_{ij}{}^2 - (a_i - a_j)^2\}(A - E) + 4(a_i - a_j)a_{ij}B = 0 .
$$

Moreover substituting the values of B^* and D^* from (5) in the first equation of (7) we get

$$
(14)\qquad 4(a_i - a_j)a_{ij}(A - E) - \{4a_{ij}{}^2 - (a_i - a_j)^2\}B = 0 .
$$

All considerations up to now are valid for any adapted frame on M. We now choose the frame at a *nonisotropic* point p which satisfies the condition $(*)$ of Lemma 1. This condition is precisely

$$
(A - E)^2 + B^2 \neq 0 .
$$

Hence (13) and (14) again lead to (12). Thus in any case $a_i = a_j$ and $a_{ij} = 0$, and the lemma is proved.

5. For proving Theorem 1 we have now arrived at the same stage as of Theorem 1 in [2]. If all point are isotropic, as is well known $H \equiv$ constant. So by analyticity if $H \not\equiv$ constant, then nonisotropic points are dense, and f is conformal by Lemma 3. Since f is conformal and H-preserving, it is easily seen that f actually preserves sectional curvature. Hence we can appeal to the main theorem of [2], which proves Theorem 1.

However we prefer to give an alternate proof which is more direct in the present case; it was suggested to us by Yau. Let Φ, Φ^* be the fundamental 2-forms of M, M^* respectively. Since f is holomorphic and conformal, we have $f^*g^* = \psi \cdot g$, and therefore $f^*\Phi^* = \Psi \cdot \Phi$, where ψ is a smooth positive real valued function. Since Φ, Φ^* are closed we get $d\psi \wedge \Phi = 0$, and since $\dim_C M \geq 2$ we see that $d\psi = 0$ or ψ is constant, i.e., f is a homothety. Since

f is H-preserving and $H \not\equiv 0$, f must be an isometry. Hence Theorem 1 is proved.

6. In this section we shall study the equivalence of conformally flat manifolds. A Riemann manifold (M^n, g) is said to be *conformally flat* if there exist a covering $\{U_\alpha\}$ of M and conformal maps $\varphi_\alpha: M \to \boldsymbol{R}^n$ where $\boldsymbol{R}^n$ is the Euclidean space with the flat metric. A natural *curvature function on a conformally flat space is the Ricci curvature, namely, if*

$$\mathscr{R} = \mathrm{Ric}\colon \tau \times \tau \to \boldsymbol{R}$$

is the Ricci tensor viewed as a bilinear form on the tangent bundle, then the Ricci curvature is the function

$$K_{\mathscr{R}}\colon G_1(M, \boldsymbol{R}) \to \boldsymbol{R}\ ,$$

where G_1 is the bundle of lines associated to τ defined by

$$K_{\mathscr{R}}([v]) = \frac{\mathrm{Ric}\,(v, v)}{\|v\|^2}\ ,$$

where $v \neq 0$ represents the line $[v]$.

If $K_{\mathscr{R}} \equiv$ constant, then as is well known, the conformally flat (M, g) is of constant curvature. So like other "good" curvature functions the local geometry of a conformally flat manifold with constant $K_{\mathscr{R}}$ is uniquely determined. On the other hand we have the following theorem.

Theorem 2. *Let $f\colon M \to M^*$ be a $K_{\mathscr{R}}$-preserving diffeomorphism. Suppose that M, M^* are conformally flat,* $\dim M \geq 3$, *and the set of points nonisotropic with respect to $K_{\mathscr{R}}$ is dense. Then f is an isometry.*

The proof is very much similar to our congruence theorem [3]. Let Sc denote the scalar curvature which is the trace of the Ricci tensor. As is well known the tensor

$$\varphi_{ij} = R_{ij} - \frac{\mathrm{Sc}}{2(n-1)} g_{ij}$$

satisfies the Codazzi's equations, i.e.,

$$(1) \qquad \varphi_{ij,k} = \varphi_{ik,j}\ .$$

As usual under the hypothesis of denseness of nonisotropic points, f is conformal. To show f is an isometry one uses (1) just as the second fundamental form in the case of hypersurfaces; cf. [3]. The interested reader can easily complete the proof.

7. In this section we shall discuss a Kählerian analogue of the theorem of Schur. We have already used the fact in Theorem 1 that if M is a connected

Kähler manifold of $\dim_C \geq 2$ and all points are isotropic with respect to H, then, H is identically constant. This is a Kählerian analogue of a theorem in Riemannian geometry due to F. Schur. Let us view this situation as follows: let $f: M \to M^*$ be a diffeomorphism of Kähler manifolds such that f carries $G_1(M, C)$ into $G_1(M^*, C)$ and

(*) there exists a real valued function $\psi: M \to \boldsymbol{R}$ such that $H^*(f_*\sigma) = \psi(p)H(\sigma)$ for each holomorphic 2-plane σ on M.

The theorem mentioned above may be formulated as follows. To say points of M^* are isotropic amounts to the (local) existence of a map $f: M \to M^*$ satisfying (*) where M is a Kähler manifold of constant holomorphic curvature $\equiv 1$. The theorem says in this case that $\psi \equiv$ constant. We now generalize this theorem as follows.

Theorem 3. *Let $f: M \to M^*$ be a diffeomorphism of connected Kähler manifolds satisfying* (*). *Suppose that* $\dim_C M \geq 2$ *and* $H \not\equiv 0$. *Then* $\psi \equiv$ *constant.*

Proof. In view of the above theorem we assume that $H \not\equiv$ constant, so that the set of nonisotropic points is dense. Then as in Theorem 1, f is conformal. The alternate argument which we gave for passing from conformal to isometry in Theorem 1 shows in the present case that f is a homothety. Hence ψ is constant.

References

[1] E. Cartan, *Les surfaces qui admettent une seconde forme fondamentale donnée,* Bull. Soc. Math. France **67** (1943) 8–32.

[2] R. S. Kulkarni, *Curvature and metric,* Ann. of Math. **91** (1970) 311–331.

[3] ——, *Congruence of hypersurfaces,* J. Differential Geometry **8** (1973) 95–102.

[4] ——, *On the Bianchi identities,* Math. Ann. **199** (1972) 175–204.

[5] ——, *On a theorem of F. Schur,* J. Differential Geometry **4** (1970) 453–456.

[6] T. Nasu, *On diffeomorphisms which preserve the sectional or Ricci curvature,* to appear.

[7] K. Nomizu & K. Yano, *Some results related to the equivalence problem,* Math. Z. **97** (1967) 29–37.

[8] I. M. Singer, *Infinitesimally homogeneous spaces,* Comm. Pure Appl. Math. **13** (1960) 685–697.

[9] S. T. Yau, *Curvature preserving diffeomorphisms,* to appear.

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